Orthopositronium Decay Spectrum using NRQED

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As noticed in Ref. [1], the Ore-Powell's classical calculation of the o-Ps$\to 3\gamma$ decay amplitude does not fulfill Low's theorem requirements for the low energy end of the photon spectrum. We reanalyze the implications of Low's theorem applied to this decay considering the interplay between the soft photon limit and the energy scales present in the e^+e^- system. For energetic photons, the spectrum agrees with the Ore-Powell result, but deviates from it when the photon energy is comparable to the positronium binding energy. In this region it is found that bound states effects are essential to reach agreement with Low's theorem and can be accounted for in the framework of non-relativistic QED.

1. Introduction

The spin-triplet (orthopositronium: o-Ps) 3γ annihilation rate was first obtained by Ore and Powell [2] in the late forties and can be found in many textbooks. Although the leptons inside the positronium atom are bound by the Coulomb potential, the effects of such binding have been commonly neglected in the calculation of the o-Ps decay rate, being the process dominated by the short distance part where the leptons annihilate into three photons. The Coulomb force will be expected to affect the annihilation process appreciably only when a photon of small momentum is involved, as already quoted in the original paper of Ore and Powell. The analysis of the low energy photon spectrum of the oPs $\to 3\gamma$ decay made in Ref. [3] takes into account these novel features by the use of the non-relativistic effective field theory methods developed to study bound states in QED and QCD [4,5,6,7].

The later work was motivated by the observation of Pestieau and Smith [1], that the Ore-Powell calculation of the o-Ps decay is in apparent contradiction with Low's theorem. While Low's theorem applied to this decay requires that the spectrum vanishes as E_γ^3, with E_γ the energy of the radiated photon, the standard calculation predicts a $\mathcal{O}(E_\gamma)$ behavior for $E_\gamma \to 0$. We show that the Ore-Powell calculation can be reconciled with Low's theorem if the latter is carefully applied considering all the energy scales present in the photon decay spectrum: the electron mass m, the binding energy of order $m\alpha^2$, and the hyperfine splitting between the singlet and triplet states of order $m\alpha^4$. The Ore-Powell computation is valid for photon energies $E_\gamma \gg m\alpha^2$. When E_γ is of order $m\alpha^2$, the Ps binding energy can not be neglected. The decay amplitude depends on a sum over an infinite set of excited Ps states, and can be written in terms of a Ps structure function which was computed in Ref. [3] using NRQED to leading order in the velocity (v) expansion. When E_γ is of order $m\alpha^4$, the decay amplitude is dominated by the o-Ps $\to$ p-Ps transition and it can be shown that the photon spectrum crosses over from an E_γ behavior above the structure function region to an E_γ^3 behavior below the p-Ps resonance region.

2. Low's theorem and o-Ps $\to 3\gamma$ decay

Low's theorem [8] gives the amplitude for soft photon emission in the scattering of charged particles. It states that the first two terms of the series expansion in powers of the photon energy of a radiative amplitude $X \to Y\gamma$ may be obtained from a knowledge of the corresponding nonradiative amplitude $X \to Y$:

$$\epsilon_\mu \mathcal{M}^\mu = \frac{\mathcal{M}_0}{k} + \mathcal{M}_1 + \mathcal{O}(k). \tag{1}$$

One finds that $\mathcal{M}_0$ and $\mathcal{M}_1$ are independent of k and completely determined from the nonradiative amplitude T_0, its derivatives in physically allowed kinematic directions, and the electromagnetic properties of the particles involved [8].

The term $\mathcal{M}_0$ arises from the emission of a photon by ingoing or outgoing charged particles and is proportional to T_0 times the universal factor $-Q_i\,\epsilon \cdot p_i/k \cdot p_i$, summed for all the external lines in the diagram. Note that if the nonradiative process involves no moving charged particles or is forbidden due to some selection rule, then $\mathcal{M}_0$ is identically zero. The term $\mathcal{M}_1$ can be expressed as a function of the magnetic moments, the amplitude T_0 and its derivatives with respect to internal variables which are not subject to constraint, e.g. the energy and angles.

Combining the amplitude behavior with that of the phase-space, the low-frequency form of the photon spectrum is

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\gamma} = \frac{A}{E_\gamma} + B + \mathcal{O}(E_\gamma), \tag{2}$$

where A is proportional to $|\mathcal{M}_0|^2$ and B is the $\mathcal{M}_0\mathcal{M}_1$ interference term. If $\mathcal{M}_0$ vanishes, the soft photon decay spectrum is of order $E_\gamma\,\mathrm{d}E_\gamma$; if both $\mathcal{M}_0$ and $\mathcal{M}_1$vanish, it is of order $E_\gamma^3\,\mathrm{d}E_\gamma$.

In the three-photon decay of o-Ps, one of the photons can have an arbitrarily small energy. The process can be then viewed as the radiative version of the o-Ps$\to 2\gamma$ decay. As the two-photon decay of o-Ps is not allowed by charge conjugation invariance, the direct application of Low's theorem yields $\mathcal{M}_{0,1} = 0$ so that the o-Ps$\to 3\gamma$ amplitude is of order $\mathcal{O}(E_\gamma)$, and the decay spectrum is

$$\frac{\mathrm{d}\Gamma_{\mathrm{oPs}\to 3\gamma}}{\mathrm{d}E_\gamma} \sim E_\gamma^3 \tag{3}$$

as $E_\gamma \to 0$. This is in contradiction with the Ore-Powell spectrum,

$$\frac{\mathrm{d}\Gamma_{3\gamma}}{\mathrm{d}x} = \frac{2m\alpha^6}{9\pi}\left[\frac{5x}{3} + \mathcal{O}(x^2)\right], \quad x = \frac{E_\gamma}{m} \tag{4}$$

which vanishes linearly with E_γ, as pointed out in Ref. [1].

To understand the origin of the contradiction, it is worth noting that in the derivation of Low's theorem, one takes the limit $E_\gamma \to 0$ and neglects all states other than those degenerate with the incoming and outgoing states, i.e. one uses $E_\gamma \ll \Delta E$, where ΔE is the energy gap to excited states. The amplitudes $\mathcal{M}_{0,1}$ depend on the charge and magnetic moment couplings between all the intermediate states degenerate with the initial or final states. A more general version of Low's theorem gives the decay spectrum for small E_γ without taking the strict $E_\gamma \to 0$ limit. One treats all states with $\Delta E \ll E_\gamma$ as degenerate states, and includes them in the computation of charge and magnetic moment matrix elements for the purposes of Low's result. Which states are included in Low's theorem then depends on the magnitude of E_γ.

In the case of Ps decay, consider the case where the photon energy is much larger than the binding energy. Then all Ps states (including o-Ps, p-Ps, radial excitations, etc.) are degenerate for the purposes of Low's theorem. In this case, there is a non-zero magnetic dipole matrix element between o-Ps and p-Ps, so that $\mathcal{M}_1$ does not vanish in this extended space of states. As a result, the decay spectrum vanishes linearly with E_γ. This is the approximation under which the Ore-Powell calculation is valid. For energies much smaller than the o-Ps–p-Ps hyperfine splitting, p-Ps as well as radial excitations are treated as excited states, the matrix element $\mathcal{M}_1$ vanishes, and the spectrum is of order E_γ^3.

The discussion above could be also applied to the 3γ annihilation of excited Ps states, like the ${}^1P_1 \to 3\gamma$ decay [9].

3. Orthopositronium Decay Amplitude

The Ore-Powell annihilation amplitude can be obtained from the free-particle decay amplitude given, to lowest order in α, by the graph in Fig. 1. If the o-Ps momentum space wavefunction is $\phi_o(\mathbf{p})$ and the free e^+e^- annihilation amplitude is $A(\mathbf{p})$, the bound-state decay amplitude to lowest order in v is

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} A(\mathbf{p})\phi_o(\mathbf{p}) \simeq A(0)\psi_o(0)\ , \tag{5}$$

where $\psi_o(0)$ is the o-Ps position space wavefunction at the origin. Spin averaging $|A|^2$ and in-

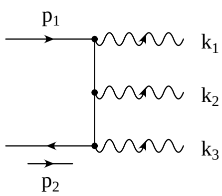

Figure 1. Three photon annihilation graph. The graph is summed over the 3! permutations of the photons.

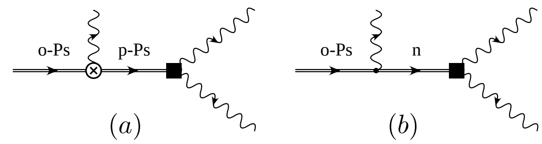

Figure 2. (a) Magnetic dipole graph for o-Ps annihilation. The solid square denotes the p-Ps annihilation vertex. (b) Electric dipole transition for o-Ps decay. The solid square denotes the $n^3P_{0,2}$ annihilation vertex

tegrating over the three-body phase-space yields the differential decay rate written in Eq. (4).

The non-relativistic effective theory computation (see details in Ref. [3]) provides a systematic way of including bound state effects in the o-Ps decay amplitude. The NRQED Hamiltonian is constructed to reproduce the QED amplitude when binding effects are neglected. Once the NRQED Hamiltonian has been determined, it can be used to compute the decay including binding corrections.

The bound state dynamics is described by the Coulomb Hamiltonian for an e^+e^- system in interaction with the quantized electromagnetic field:

$$\begin{aligned} H &= H_0 + H_{\rm int} \\ H_0 &= \frac{\mathbf{p}^2}{m} - \frac{\alpha}{r} \\ H_{\rm int} &= -\frac{e}{2m}\left[\boldsymbol{\sigma}_\phi + \boldsymbol{\sigma}_\chi\right]\cdot\mathbf{B} - e\,\mathbf{x}\cdot\mathbf{E} \end{aligned} \tag{6}$$

with $\mathbf{x}$ and $\mathbf{p}$ the relative position and relative momentum of the pair of leptons, and $\boldsymbol{\sigma}_\phi$, $\boldsymbol{\sigma}_\chi$ the Pauli matrices acting on the electron and positron spinors. The electromagnetic interactions are through multipole interactions with the electric and magnetic fields and only the electric and magnetic dipole interactions are shown in Eq. (6), the higher multipoles being of higher order in v.

The Coulomb Hamiltonian H_0 is the leading term in the velocity power counting. The kinetic energy and Coulomb potential are of the same order in v. The energies and wavefunctions of H_0 are thus the Coulomb wavefunctions with reduced mass $m/2$. The electric and magnetic dipole interaction terms are treated as perturbations. The o-Ps decay amplitude has both short distance and long-distance contributions. The long distance part of the o-Ps decay amplitude entails the radiation of the soft photon ($E_\gamma \ll m$) from the o-Ps bound state, whereas the short distance part is given by the on-shell $e^+e^- \to 2\gamma$ QED decay amplitude where the photons are hard ($E \sim m$).

The effective theory graphs which describe the o-Ps$\to 3\gamma$ decay are shown in Fig. 2. The magnetic term in $H_{\rm int}$ can induce a $1\,{}^3S_1 \to 1\,{}^1S_0$ transition between Coulomb e^+e^- states. The only allowed intermediate state in the dipole approximation is the p-Ps ground state, with energy E_p, so that $E_o - E_p = \Delta E_{\rm hfs}$. The electric dipole term $-e\,\mathbf{x}\cdot\mathbf{E}$ in $H_{\rm int}$ can change orbital angular momentum by one unit, allowing for transitions from ground state o-Ps to $n\,{}^3P_{0,2}$ states ($n \neq 1$).

To compute the o-Ps decay amplitude matching coefficient, one compares the effective theory result with the QED computation which neglects bound state effects. For the matching computation, the energy separation between o-Ps and the intermediate states, $E_o - E_n \sim \mathcal{O}(m\alpha^2)$ is taken to be much smaller than the photon energy E_γ.

4. Results for the low energy spectrum

The o-Ps$\to 3\gamma$ decay spectrum from our effective theory is written in terms of the magnetic and electric amplitudes obtained from the graphs in Fig. 2 [3],

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}x} = \frac{m\alpha^6}{9\pi}x\left[|a_m|^2 + \frac{7}{3}|a_e|^2\right]. \tag{7}$$

For the magnetic term a_m, it is found

$$a_m = \frac{E_\gamma}{E_\gamma - \Delta E_{\rm hfs} - i\Gamma_p/2}, \tag{8}$$

$$\Delta E_{\rm hfs} = \frac{7}{12}\, m\,\alpha^4 \quad , \quad \Gamma_p = \frac{1}{4} m\alpha^5. \tag{9}$$

The electric amplitude is written in terms of the p-wave Coulomb Green's function [3], which contains the sum over $n^3P_{0,2}$ states in Fig. 2b:

$$a_e(E_\gamma) = \frac{4\pi E_\gamma}{\psi_o(0)} \int_0^\infty {\rm d}y\, y^4\, G_1(0, y; E_\gamma)\, \psi_o(y). \tag{10}$$

A closed analytical formula for this integral has been given in Ref. [10]. The magnetic and electric amplitudes behave as

$$a_m(E_\gamma) \;=\; \begin{cases} 1 & E_\gamma \gg \Delta E_{\rm hfs}, \\ -\frac{E_\gamma}{\Delta E_{\rm hfs}} & E_\gamma \ll \Delta E_{\rm hfs}. \end{cases}$$
$$a_e(E_\gamma) \;=\; \begin{cases} 1 & E_\gamma \gg m\alpha^2, \\ \frac{2E_\gamma}{m\alpha^2} & E_\gamma \ll m\alpha^2. \end{cases} \tag{11}$$

In the limit $E_\gamma \gg m\alpha^2$ the Ore-Powell spectrum (Eq. (4)) is recovered, and it is shown that the sum of the magnetic and electric dipole transitions in the effective theory gives the full theory amplitude at leading order in the non-relativistic expansion. The matching condition, which is the difference of the two results, vanishes [3] and there is no additional three-photon annihilation term in the NRQED Hamiltonian.

The ratio of the photon spectrum to the Ore-Powell value is shown in Fig. 3 up to $E_\gamma \sim m\alpha/2$. At energies large compared with the binding energy, a_e and a_m approach their values in Eq. (11), and the spectrum reproduces the QED one. The magnetic and electric dipole terms contribute in the ratio 3 : 7. At energies small compared with the binding energy, the electric dipole transitions decouple, and one is left with the magnetic term due to the p-Ps resonance contribution, of 3/10 of the Ore-Powell value. At energies much smaller than the hyperfine splitting, the p-Ps state also decouples, and the decay rate vanishes as E_γ^3, as predicted by Low's theorem.

The results are consistent with the Ore-Powell spectrum and with Low's theorem. They include binding effects in a systematic expansion in powers of v.

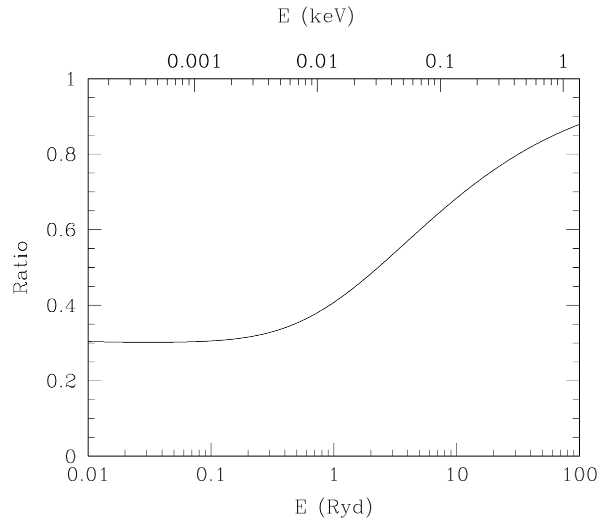

Figure 3. Ratio of the o-Ps decay spectrum including binding energy corrections to the Ore-Powell spectrum (1 Ryd= $m\alpha^2/2$).

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REFERENCES

1. J. Pestieau and C. Smith, Phys. Lett. B524 (2002) 395.
2. A. Ore and J.L. Powell, Phys. Rev. 75 (1949) 1696.
3. A. V. Manohar and P. Ruiz-Femenia, Phys. Rev. D **69** (2004) 053003.
4. W. E. Caswell and G. P. Lepage, Phys. Lett. B **167**, 437 (1986).
5. G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995) [Erratum-ibid. D **55**, 5853 (1997)].
6. M. E. Luke, A. V. Manohar and I. Z. Rothstein, Phys. Rev. D **61**, 074025 (2000).
7. A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. **64**, 428 (1998).
8. F. E. Low, Phys. Rev. **110**, 974 (1958).
9. D. Gromes, Eur. Phys. J. C **36** (2004) 169.
10. M. B. Voloshin, Mod. Phys. Lett. A **19** (2004) 181.